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Effect of Boundary Condition Description on Convergence of Solution in a Boundary-Value Problem

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ABSTRACT

It is well known that the numerical accuracy of a series solution to a boundary-value problem by the direct method depends on the technique of approximate satisfaction of the boundary conditions and on the stage of truncation of the series. On the other hand, it does not appear to be generally recognized that, when the boundary conditions can be described in alternative equivalent forms, the convergence of the solution is significantly affected by the actual form in which they are stated. The importance of the last aspect is studied for three different techniques of computing the deflections of simply supported regular polygonal plates under uniform pressure. It is also shown that it is sometimes possible to modify the technique of analysis to make the accuracy independent of the description of the boundary conditions.

INTRODUCTION

In any boundary-value problem, a solution is sought which satisfies the governing differential equation and boundary conditions of the problem. Where an exact solution is not available, one chooses approximate solutions. In the direct (or boundary) method of approximate analysis, a series which satisfies the differential equation exactly at all stages of truncation is chosen and the boundary conditions are satisfied approximately. It is well known that the accuracy of such a solution depends on the technique of approximate satisfaction of the boundary conditions and on the stage of truncation of the series. On the other hand, it does not appear to be generally recognized that, when the boundary conditions can be described in alternative equivalent forms, the convergence of the solution is significantly affected by the actual form in which they are stated [1]. In this paper, the importance of the last aspect is studied for three different techniques (collocation, Taylor expansion, and successive integration of boundary errors) of analyzing the deflection of simply-supported regular polygonal plates under uniform transverse load. It is also shown that one may be able to avoid the difficulty of identifying the "best description" by evolving a special procedure which eliminates the influence of the parameter generating the alternative descriptions.

Series Solution for a Polygonal Plate

Consider a regular polygonal plate with n sides and n axes of symmetry, subjected to a uniform transverse pressure q as shown in Fig. 1. a, the radius of the inscribed circle serves as a characteristic length. The flexural stiffness of the plate is D [2].

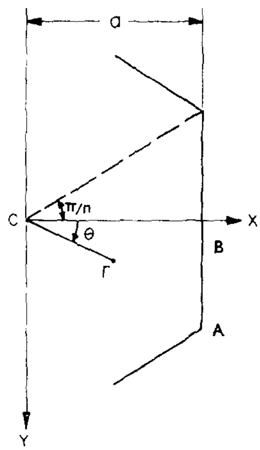

Fig. 1. Coordinate system for regular polygon.

The deflection function w for the middle surface of the plate should satisfy the differential equation

$$D \cdot \nabla^4 w = q \tag{1}$$

or, in terms of the nondimensional deflection parameter $\bar{w} = Dw/qa^4$,

$$\nabla^4 \bar{w} = 1/a^4 \tag{1a}$$

and also the boundary conditions on the edges.

Taking advantage of the n-fold cyclic symmetry, a series solution for Eq. (1a) can be conveniently written, in polar coordinates (r, θ) as

$$\bar{w} = r^4/64a^4 + \sum_{m=0}^{M-1} (A_m + B_m r^2)\, r^{mn} \cdot \cos mn\theta \tag{2}$$

where A_m, B_m are unknown parameters to be determined by satisfying the boundary conditions on the straight edges and M represents the stage of truncation of the series or the order of approximation.

Statement of Boundary Conditions for Simply Supported Edges

The normal and tangential moments on the edges are given [2] by $M_n = -D(w_{nn} + \nu w_{tt})$ and $M_t = -D(w_{tt} + \nu w_{nn})$ where ν is the Poisson's ratio of the material. Then, the boundary conditions for a simply supported edge, say $AB(x = a)$, are

$$\bar{w} = 0 \tag{3}$$

and

$$M_x = -qa^4(\bar{w}_{xx} + \nu\bar{w}_{yy}) = 0$$

or

$$\bar{w}_{xx} + \nu\bar{w}_{yy} = 0 \tag{4}$$

As the edge AB is rectilinear, using the first condition (3), the second boundary condition (4) can be written in one of the alternative forms,

$$\bar{w}_{xx} = 0 \qquad (\text{implying } M_x - \nu M_y = 0) \tag{4a}$$

and

$$\nabla^2\bar{w} = 0 \qquad (\text{implying } M_x + M_y = 0) \tag{4b}$$

or, more generally, as

$$\bar{w}_{xx} + K\bar{w}_{yy} = 0 \qquad (\text{implying } M_x + K_1 M_y = 0), \tag{4c}$$

where K is *any* arbitrary constant and $K_1 = (K - \nu)/(1 - \nu K)$.

The forms of the governing differential equation (1a) and the boundary conditions (3) and (4), clearly show that, for simply supported rectilinear plates, $\bar{w}$ is independent of ν. Obviously, this statement is not applicable if the plate boundary is curvilinear—circular or otherwise—in part or in full.

If an exact solution were feasible, the actual value attributed to K in the general form, Eq. (4c), is immaterial. But when an approximate series solution is sought by a boundary method, the boundary errors may be expected to depend on the specific value of K chosen.

Substitution of the deflection function (2) into Eqs. (3) and (4c) yields the boundary error equations,

$$\left[r^4/64a^4 + \sum_{m=0}^{M-1} (A_m + B_m r^2)\, r^{mn} \cos mn\theta\right]_{x=a} = 0 \tag{5}$$

and

$$[r^2\{(1+3K)+2(1-K)\cos^2\theta\}/16a^4 + (1-K)\sum_{m=0}^{M-1} A_m mn(mn-1)\, r^{mn-2}\cos(mn-2)\theta + \sum_{m=0}^{M-1} B_m(mn+1)\, r^{mn}\{2(1+K)\cos mn\theta + (1-K)\, mn\cos(mn-2)\theta\}]_{x=a} = 0. \tag{6}$$

Three Methods for Satisfaction of the Boundary Conditions

For purposes of the present study, three different methods, viz. (a) collocation, (b) Taylor expansion, and (c) successive integration of boundary errors, are applied for the satisfaction of the error Eqs. (5) and (6).

(a) *Collocation*: In the simple collocation procedure, the boundary conditions (5), (6) are satisfied at discrete points. For convenience, a suitable number of equidistant points are chosen in the semi-edge AB. For a convergence study, the number of such points is increased successively.

(b) *Taylor expansion*: In the Taylor expansion procedure, one sets to zero the first few tangential derivatives of the boundary errors at B. This is conveniently achieved by the following procedure.

The errors are expressed in a polynomial series in y and Eqs. (5) and (6) are written as

$$(a^4+2a^2y^2+y^4)/64a^4 + \sum_{m=0}^{M-1} a^{mn} A_m \sum_{p=0}^{P} (-1)^p C_{2p}^{mn} a^{-2p} y^{2p} + \sum_{m=0}^{M-1} a^{mn+2} B_m \sum_{p=0}^{P} (-1)^p (C_{2p}^{mn} - C_{2p-2}^{mn})\, a^{-2p} y^{2p} = 0, \tag{7a}$$

$$[(3+K)a^2 + (1+3K)y^2]/16a^4 + (1-K)\sum_{m=0}^{M-1} A_m mn(mn-1)\, a^{mn-2} \sum_{p=0}^{P-1} (-1)^p C_{2p}^{mn-2} a^{-2p} y^{2p} + \sum_{m=0}^{M-1} B_m (mn+1)\, a^{mn}\Big[2(1+K)\sum_{p=0}^{P}(-1)^p C_{2p}^{mn} a^{-2p} y^{2p} + (1-K)\, mn \sum_{p=0}^{P-1} (-1)^p \{C_{2p}^{mn-2} - C_{2p-2}^{mn-2}\}\, a^{-2p} y^{2p}\Big] = 0, \tag{7b}$$

where P represents the integer $mn/2$ or $(mn-1)/2$ as appropriate.

The aggregate coefficient of each power of y, in each of the equations, is set equal to zero. Thus one develops the following twin systems of simultaneous equations in A_m and B_m

$$\sum_{m=0}^{M-1} a^{mn} A_m C_{2p}^{mn} + \sum_{m=0}^{M-1} a^{mn+2} B_m (C_{2p}^{mn} - C_{2p-2}^{mn}) = (-1)^{p+1} C_p^2/64 \qquad (p = 0, 1, 2, \ldots) \tag{8a}$$

and

$$(1 - K) \sum_{m=0}^{M-1} a^{mn-2} A_m mn(mn - 1)\, C_{2p}^{mn-2}$$
$$+ \sum_{m=0}^{M-1} a^{mn} B_m (mn + 1)[2(1 + K)\, C_{2p}^{mn} + (1 - K)\, mn\{C_{2p}^{mn-2} - C_{2p-2}^{mn-2}\}]$$
$$= (-1)^{p+1} a^{2p} \{(3 + K)\, C_{p+1}^{1} a^2 + (1 + 3K)\, C_{p-1}^{1}\}/16\mathrm{a}^4 \qquad (p = 0, 1, 2, \ldots). \tag{8b}$$

It may be noted that the R.H.S. in Eqs. (8a) and (8b) are zero when $p > 2$. For the sth-order approximation $M = s + 1$.

(c) *Successive integration*: In this method, the error equations (5) and (6) are successively integrated along the boundary to obtain the requisite set of simultaneous equations in the arbitrary parameters A_m, B_m. The sth pair of equations in the set are

$$\int_0^t \cdots \int_0^y \int_0^y \left[(r^4/64a^4) + \sum_{m=0}^{M-1} (A_m + B_m r^2)\, r^{mn} \cos mn\theta\right]_{x=a} (dy)^s$$
$$= [a^4(t^s/s!) + 4a^2 t^{s+2}/(s + 2)! + 24 t^{s+4}/(s + 4)!]/64a^4$$
$$+ \sum_{m=0}^{M-1} a^{mn} A_m \sum_{p=0}^{P} (-1)^p C_{2p}^{mn} a^{-2p} t^{2p+s} (2p)!(2p + s)!$$
$$+ \sum_{m=0}^{M-1} a^{mn} B_m \sum_{p=0}^{P} (-1)^p \cdot C_{2p}^{mn} a^{-2p} [a^2 t^{2p+s} (2p)!/(2p + s)!$$
$$+ t^{2p+s+2}(2p + 2)!/(2p + s + 2)!] = 0 \tag{9}$$

and

$$\int_0^t \cdots \int_0^y \int_0^y \Big[r^2\{(1+3K)+2(1-K)\cos^2\theta\}/16a^4$$

$$+(1-K)\sum_{m=0}^{M-1} A_m mn(mn-1)r^{mn-2}\cos(mn-2)\theta$$

$$+\sum_{m=0}^{M-1} B_m(mn+1)r^{mn}\{2(1+K)\cos mn\theta$$

$$+(1-K)\,mn\cos(mn-2)\theta\}\Big]_{x=a}(dy)^s$$

$$=[(3+K)\,a^2t^s/s!\;+\;2(1+3K)\,t^{s+2}/(s+2)!]/16\mathrm{a}^4$$

$$+(1-K)\sum_{m=0}^{M-1} A_m mn(mn-1)\,a^{mn-2}$$

$$\times\sum_{p=0}^{P-1}(-1)^p C_{2p}^{mn-2}a^{-2p}t^{2p+s}(2p)!/(2p+s)!$$

$$+\sum_{m=0}^{M-1} B_m(mn+1)\,a^{mn}\Big[2(1+K)\sum_{p=0}^{P}(-1)^p C_{2p}^{mn}a^{-2p}t^{2p+s}(2p)!/(2p+s)!$$

$$+(1-K)\,mn\sum_{p=0}^{P-1}(-1)^p C_{2p}^{mn-2}\{a^{-2p}t^{2p+s}(2p)!/(2p+s)!$$

$$+a^{-2p-2}t^{2p+s+2}(2p+2)!/(2p+s+2)!\}\Big]=0, \tag{10}$$

where $t=\tan(\pi/n)$.

An sth-order approximation is obtained from this set of simultaneous equations by going up to $M=s$.

Convergence Studies

Numerical analysis is carried out for different orders of regular polygons of a material with $\nu=0.3$. Polygons of up to 100 sides in some cases and of up to 15 sides in other cases are investigated. Convergence of solution is studied for four values of K, viz. 0, 0.3, 1.0, and 1.5. It is of interest to note that 0.3 represents the physical conditions of zero normal edge moment ($M_n=0$ on $x=a$) and 1.0 represents the physical condition of zero sum of normal and tangential edge moments ($M_n+M_t=0$ on $x=a$). For all the cases, the unknown parameters,

A_m, B_m are determined for various orders of approximation ($M = 2, 3, \ldots, 9$). The value of A_0 yields the central deflection parameters $\bar{w}_c$ for the plate. The convergence of this value as a function of K, n and M is studied for the three methods.

Collocation

Results obtained by the collocation procedure are shown in Fig. 2. The trends in this figure would indicate that (a) with $K \leqslant 1$, the solution converges from

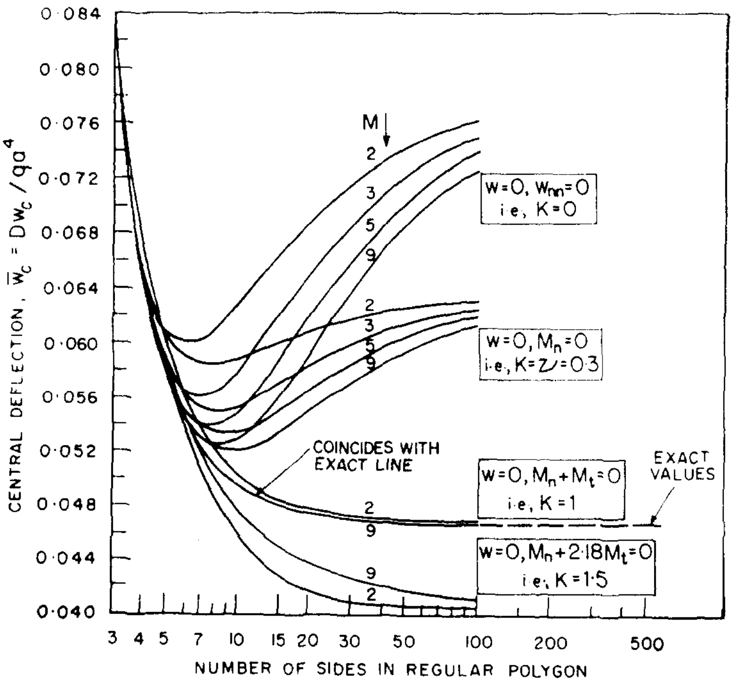

FIG. 2. Convergence of collocation procedure with different values of K: central deflections.

above, the convergence becoming slower with increasing number of sides; (b) with $K > 1$, the solution converges from below, the convergence again worsening with increasing number of sides; (c) the convergence improves as $K \rightarrow 1$ from either side; and (d) $K = 1$ yields very rapid convergence so that a 9-term solution is highly accurate even for many-sided polygons with a hundred or more sides. By comparison of the results for $K = 1$, with the exact values which are evaluated from [3] and given in Table I, it is found that the 2-term solution is in error by 3.5 % for

TABLE I

ACCURACY OF APPROXIMATE SOLUTIONS WITH $K = 1$: COMPARISON WITH EXACT VALUES EVALUATED FROM [3]

n	Collocation 2 terms	Collocation 9 terms	Successive Integration 2 terms	Successive Integration 8 terms	Exact Values [3]
4	0.067000	0.064998	0.065086	0.064998	0.064998
5	0.060199	0.058070	0.058191	0.058069	0.058067
6	0.056450	0.054575	0.054678	0.054572	0.054571
7	0.054120	0.052531	0.052610	0.052526	0.052525
8	0.052560	0.051220	0.051280	0.051215	0.051214
10	0.050656	0.049683	0.049718	0.049678	0.049677
15	0.048656	0.048151	0.048162	0.048147	0.048147
40	0.047148	0.047063	—	—	0.047061
100	0.046920	0.046906	—	—	0.046905

6 sides, 1.1 % for 15 sides and 0.04 % for 100 sides, while the errors in the 9-term solution are as small as 0.011 % for 6 sides, 0.0083 % for 15 sides and 0.0021 % for 100 sides. The corresponding errors using the $M_n = 0$ condition ($K = 0.3$) are as high as 8.5 %, 24.9 %, 34.7 %, for $M = 2$ and 0.36 %, 11.4 %, 30.9 % for $M = 9$.

Successive Integration

The results by the successive integration procedure are presented in Fig. 3. The value of K is found to have the same influence on convergence as in the collocation procedure except that the absolute accuracy of the successive integration method is found to be significantly superior (see Tables I and II or compare Figs. 2 and 3). With $K = 1$, the errors in $\bar{w}_c$ by a 2-term solution are only 0.18 % for $n = 6$ and 0.02 % for $n = 15$.

Taylor Expansion

The convergence trends for $\bar{w}_c$ obtained for a hexagon ($n = 6$) with the four values of K are presented in Table II. In the studies up to $M = 9$, the results oscillate unsatisfactorily when $K = 0$, 0.3 and 1.5. On the other hand, with $K = 1$, $\bar{w}_c$ does decrease monotonically, and rapidly towards the exact value. Similar trends were observed for the other values of n investigated (upto $n = 100$). As in the other two procedures, the convergence, in general, deteriorates with increasing n.

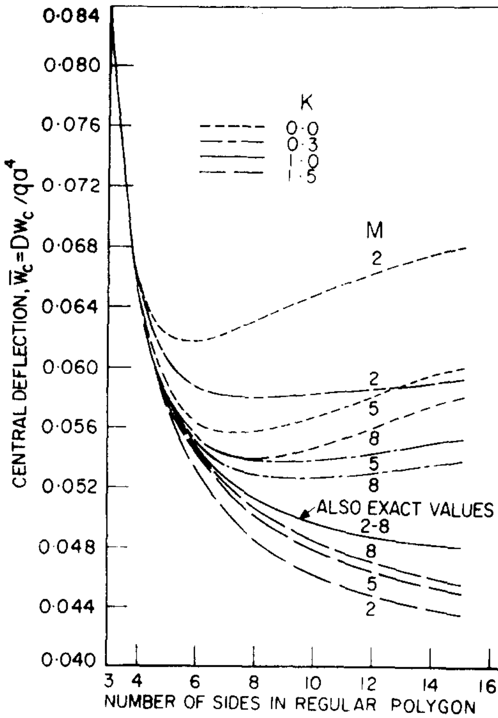

FIG. 3. Convergence of successive integration procedure with different values of K: central deflections.

TABLE II

DATA TO CONFIRM EFFECT OF K ON CONVERGENCE
APPROXIMATIONS FOR CENTRAL DEFLECTION $\bar{w}_c$ FOR A HEXAGON ($n = 6$), EXACT VALUE BEING 0.054571

1	2	3	4	5	6	7	8
	TAYLOR EXPANSION					Collocation	Successive Integration
M \ K	0	0.3	1.0	1.5	Implicitly Eliminated	1.0	1.0
2	0.06823	0.06054	0.05336	0.05455	0.04896	0.05645	0.0546780
3	0.06931	0.05946	0.05412	0.05323	0.05359	0.05479	0.0545806
4	0.10366	0.05955	0.05434	0.05306	0.05418	—	0.0545743
5	0.04023	0.06103	0.05443	0.05310	0.05436	0.05460	0.0545728
6	0.04963	0.06966	0.05448	0.05317	0.05444	—	0.0545724
7	0.05193	0.03914	0.05450	0.05325	0.05448	—	0.0545722
8	0.05290	0.05038	0.05452	0.05333	0.05451	—	0.0545721
9	0.05341	0.05237	0.05453	0.05339	0.05452	0.05458	—

Thus the results from all three procedures, those from the Taylor expansion procedure most emphatically of all, lead to an important conclusion. The choice of a value for the arbitrary constant K in the description of the boundary conditions has very substantial influence on the convergence of the solution—the apparently natural value of 0.3 yield slow convergence, whereas the value of 1.0, which appears artificial at first sight, yields the best convergence.

A Method of Eliminating the Influence of K

From the above discussion, it is clear that the convergence of the solution is best for one particular description of the boundary condition among a full spectrum of alternatives. Most times, it is not easy to identify such a "best description" by physical reasoning so that an extensive numerical investigation is needed. The question arises whether one cannot really eliminate the influence of the arbitrary parameter involved so as to sidestep the question of identifying the best value for the parameter. In the problem under consideration such a procedure is possible when applying the Taylor expansion method. The solution can be made independent of K by a judicious selection of the sequence of simultaneous equations to be solved for A_m, B_m.

Consider the sets of Eqs. (8a) and (8b). Choosing the first two equations from (8a) and only the first from (8b) one has

$$\sum_m a_{mn} A_m + \sum_m a^{mn+2} B_m = -1/64, \tag{11a}$$

$$\sum_m a_{mn} C_2^{mn} A_m + \sum_m a^{mn+2} (C_2^{mn} - 1) B_m = 1/32, \tag{11b}$$

and

$$(1 - K) \sum_m a^{mn-2} mn(mn - 1) A_m$$
$$+ \sum_m a^{mn} (mn + 1)\{2(1 + K) + (1 - K)\, mn\} B_m = -(3 + K)/16a^2. \tag{11c}$$

One notices that they form a set given by

$$\bar{w} = 0 \tag{12a}$$

[from (8a)]

$$\bar{w}_{yy} = 0 \tag{12b}$$

and

$$\bar{w}_{xx} + K\bar{w}_{yy} = 0 \qquad \text{[from (8b)]} \tag{12c}$$

Because of Eqs. (12b) and (12c), it is obvious that the solution to this set of three simultaneous equations is independent of the parameter K. The same is easily shown to be true for any solution obtained by using the first M equations from the set (8a) and first $(M - 1)$ equations from the set (8b). Hence it can be concluded that, if the Taylor expansion method is modified by taking the first M equations from (8a) and only the first $(M - 1)$ equations from (8b) and solving these $(2M - 1)$ equations for $(A_0, A_1, ..., A_{M-1}, B_0, B_1, ..., B_{M-2})$, the resulting solution at any stage of approximation M, is independent of K. The success of this procedure is evident from col. 6 Table II (for $n = 6$).

Significance of Superiority of Results with $K = 1$

It has been noted that convergence of solutions is best with $K = 1$. One would naturally like to have an explanation for this. To appreciate the problem, it may be worth digressing and examining Kirchoff's treatment of a free edge [2]. On such an edge, from physical considerations, one can specify three independent boundary conditions viz., $M_n = 0$, $Q_n = 0$ and $M_{nt} = 0$. However, due to a mathematical inadequacy in the thin plate formulation, one can satisfy only two independent conditions. To do so, one may coalesce $Q_n = 0$ and $M_{nt} = 0$ into a single condition $(Q_n + K\, \partial M_{nt}/\partial t) = 0$ where K is any arbitrary constant. From a variational approach, Kirchoff arrived at unity as the most appropriate value for K. Thomson and Tait confirmed the appropriateness of this by physical reasoning.

In the case of a simply supported straight edge, one would readily see that there are again three independent homogeneous conditions $\bar{w} = 0$, $M_n = 0$, $M_t = 0$ whereas only two can be explicitly stated and satisfied. Hence, following the analogy with the free edge, one would coalesce $M_n = 0$ and $M_t = 0$ into the single condition $(M_n + KM_t) = 0$. The present numerical investigation has shown that unity is the most appropriate value for K. That is, it is best to describe the edge moment condition as the zero condition for the variant $(M_n + M_t)$. It would be instructive to investigate the physical and mathematical reasons for this result.

Computational Accuracy of Data

The collocation and Taylor expansion solutions were programmed in autocode and computed on a Ferranti Sirius computer working with 8 significant figures. The successive integration procedure was programmed in FORTRAN IV and solved on a CDC-3600 Computer working with 15 significant figures. It has been checked that, in all the final results, roundoff errors occur only beyond the fifth significant figure.

Concluding Remarks

With an example, it has been demonstrated that, where alternative equivalent descriptions of boundary conditions are possible, satisfactory and reliable convergence of the series solution by the direct method may depend upon the choice of the boundary condition. Two significant conclusions in respect of thin-plate theory may be drawn as incidental to this study. Corresponding to the Kirchoff conditions for free edges, one has the modified conditions $\bar{w} = 0$, $\nabla^2\bar{w} = 0$ to satisfactorily represent simple supports for rectilinear edges. The experience with free edges and simply supported straight edges would suggest that proper descriptions of boundary conditions for different types of supports may need further investigation from the point of view of convergence of series solutions.

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